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EXACT REGULARITY OF $\bar\partial$ ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. We show there is a solution operator to $\bar\partial$ which is bounded as a map $W^s_{(0,1)}(\Omega)\cap\ker\bar\partial\to W^s(\Omega)$ for all $s\geq 0$.

1. INTRODUCTION

Let $\Omega\subset\mathbb{C}^n$ be a smooth, bounded weakly pseudoconvex domain. In [6] Hörmander showed that given $f\in L^2_{p,q}(\Omega)$, $p,q\leq n$, such that $\bar\partial f=0$, one can find a solution $u\in L^2_{p,q-1}(\Omega)$ with $\bar\partial u=f$. We refer to the problem of finding a solution u given $\bar\partial$-closed data f as the $\bar\partial$-problem.

Regularity of the canonical solution, the solution of minimal L^2-norm, in terms of regularity of the data form was undertaken by Kohn. Let $W^s_{(p,q)}(\Omega)$ be the Sobolev space consisting of (p,q)-forms with components which are functions whose derivatives of order $\leq s$ are in $L^2(D)$. When Ω is a strictly pseudoconvex domain it was shown in [8, 9] that there is a solution operator to the $\bar\partial$-problem which maps $W^s_{(p,q)}(\Omega)\to W^s_{(p,q-1)}(\Omega)$ continuously.

When Ω is weakly pseudoconvex, Kohn, in [10], showed that given any $s>0$, one can find a solution operator which maps $W^s_{(p,q)}(\Omega)\to W^s_{(p,q-1)}(\Omega)$ continuously but the solution operator depends on the Sobolev level s. This was accomplished by working with weighted L^2 spaces, so u is not the canonical solution. In fact, as shown by Barrett [1] and Christ [4], in general such regularity (the property of $u\in W^s_{(p,q-1)}(\Omega)$ if $f\in W^s_{(p,q)}(\Omega)$, with estimates) is not exhibited by the canonical solution; there exists a smoothly bounded (weakly) pseudoconvex domain and a $\bar\partial$-closed form $f\in C^\infty_{(0,q)}(\overline{D})$ such that the canonical solution to $\bar\partial u=f$ is not in $C^\infty_{(0,q-1)}(\overline{D})$.

The question of whether there exists a solution operator (necessarily not producing the canonical solution) which continuously maps $W^s_{(p,q)}(\Omega)\to W^s_{(p,q-1)}(\Omega)$, *for all s* simultaneously, suggests itself (see the discussion in Section 5.2 in [13]). Straube, in [13], produces for each $\delta>0$ a linear solution operator, denoted here and later in this paper as $S_{-\delta}:W^s_{(p,q)}(\Omega)\to W^{s-\delta}_{(p,q-1)}(\Omega)$ for all $s\geq 1$.

Our work in this paper relates to the question of whether one can take $\delta=0$ and still obtain an operator analogous to Straube's operator above. In [5], the author showed that such a solution operator exists (for $s>1/2$) under the assumption of the existence of a solution operator to the

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boundary problem (the $\bar{\partial}_b$-problem) with similar regularity properties. As the methods involved included a reduction to the boundary, it is not surprising that $\bar{\partial}_b$ should arise.

In this paper we show that a solution operator can be constructed independent of $\bar{\partial}_b$. This is possible in dimension 2, as was the work done in [5]. Our solution relies on Straube's; we produce an approximate solution whose error terms lie in $W^{s+1}(\Omega)$ so that Straube's operator can be used to correct for them. We thus present our Main Theorem:

Main Theorem. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. There exists a solution operator K such that $\bar{\partial}Kf = f$ for all $f \in L^2_{(0,1)}(\Omega) \cap \ker \bar{\partial}$ and $K : W^s_{(0,1)}(\Omega) \cap \ker \bar{\partial} \to W^s(\Omega)$ continuously for all $s \geq 0$.*

Our techniques are based on a reduction to the boundary, and we rely on the setup presented in [2] as well as a microlocalization similar to that in [11].

2. Some background

We refer to [14] for some basics on pseudodifferential operators. We describe here how they will be used in this article. We will use the technique of reducing our boundary value problem to the boundary, as in [2]. The boundary will be covered by neighborhoods on each of which there is a coordinate chart with which we will express the operators in the resulting equations on the boundary. We express these as pseudodifferential operators.

Let D be a differential operator on $\partial\Omega$. Let χ_j be such that $\{\chi_j \equiv 1\}_j$ is a covering of $\partial\Omega$. And let φ_j be a partition of unity subordinate to this covering. Locally, we describe D in terms of its symbol, $\sigma(D)$ according to

$$Dg = \frac{1}{(2\pi)^4}\int \sigma(D_j)(x,\xi)\widehat{\chi_j g}(\xi)d\xi$$

on supp φ_j, where D_j is a local expression of the operator D_j on a coordinate patch in a neighborhood of supp χ_j. For such coordinate patches we will take the defining function ρ as one of the coordinates. In what follows, we shall drop the subscript j on the operator D_j, as they stem from the same operator, but keep in mind we work with such local expressions of operators. Then we can describe the operator D globally on all of Ω by

$$Dg = \frac{1}{(2\pi)^4}\sum_j \varphi_j \int \sigma(D)(x,\xi)\widehat{\chi_j g}(\xi)d\xi. \tag{2.1}$$

In our description of a solution we further use the following microlocalization of the transform space into three regions, following [3, 7, 11, 12]. We write $\xi_{1,2} := (\xi_1, \xi_2)$, and define the three

regions

$$\mathcal{C}^+ = \left\{\xi \middle| \xi_3 \geq \frac{1}{2}|\xi_{1,2}|,\ |\xi| \geq 1\right\}$$
$$\mathcal{C}^0 = \left\{\xi \middle| -\frac{3}{4}|\xi_{1,2}| \leq \xi_3 \leq \frac{3}{4}|\xi_{1,2}|\right\} \cup \{\xi | |\xi| \leq 1\}$$
$$\mathcal{C}^- = \left\{\xi \middle| \xi_3 \leq -\frac{1}{2}|\xi_{1,2}|,\ |\xi| \geq 1\right\}.$$

Associated to the three regions we define the functions $\psi^+(\xi)$, $\psi^0(\xi)$, and $\psi^-(\xi)$ with the following properties: $\psi^+, \psi^0, \psi^- \in C^\infty$, are symbols of order 0 with values in $[0,1]$, ψ^+ (resp. ψ^0, resp. ψ^-) restricted to $|\xi| = 1$ has compact support in $\mathcal{C}^+ \cap \{|\xi| = 1\}$ (resp. $\mathcal{C}^0 \cap \{|\xi| = 1\}$, resp. $\mathcal{C}^- \cap \{|\xi| = 1\}$) with $\psi^-(\xi) = \psi^+(-\xi)$ and ψ^0 is given by $\psi^0(\xi) = 1 - \psi^+(\xi) - \psi^-(\xi)$. Furthermore for $|\xi| \geq 1$, $\psi^-(\xi) = \psi^-\left(\frac{\xi}{|\xi|}\right)$ (resp. $\psi^0(\xi) = \psi^0\left(\frac{\xi}{|\xi|}\right)$, $\psi^+(\xi) = \psi^+\left(\frac{\xi}{|\xi|}\right)$). The relation $\psi^0(\xi) + \psi^+(\xi) + \psi^-(\xi) = 1$ is to hold on all of $\mathbb{R}^3$. Due to the radial extensions from the unit sphere, the functions $\psi^-(\xi)$, $\psi^0(\xi)$, and $\psi^+(\xi)$ are symbols of zero order pseudodifferential operators. The operator Ψ^+ (resp. Ψ^-) is defined as the operator with symbol ψ^+ (resp. ψ^-). We do not have need for the operator defined by the symbol ψ^0 and as the above notation would conflict with our notations of generic pseudodifferential operators of order 0, we have left out this operator.

The support of ψ^0 is contained in $\mathcal{C}^0$, and from the above requirements we have the support of ψ^+ (resp. ψ^-) is contained in $\mathcal{C}^+ \cup \{|\xi| \leq 1\}$ (resp. $\mathcal{C}^- \cup \{|\xi| \leq 1\}$). We make the further restrictions that the supports of ψ^+ and ψ^- are contained in conic neighborhoods; we define

$$\widetilde{\mathcal{C}}^+ = \left\{\xi \middle| \xi_3 \geq \frac{1}{2}|\xi_{1,2}|\right\}$$
$$\widetilde{\mathcal{C}}^- = \left\{\xi \middle| \xi_3 \leq -\frac{1}{2}|\xi_{1,2}|\right\}.$$

We then assume that the support of ψ^+ and ψ^- are contained in $\widetilde{\mathcal{C}}^+$ and $\widetilde{\mathcal{C}}^-$, respectively, such that the restrictions, $\psi^+\big|_{\{|\xi| \leq 1\}}$ and $\psi^-\big|_{\{|\xi| \leq 1\}}$ have support which is relatively compact in the interior of the regions $\widetilde{\mathcal{C}}^+$ and $\widetilde{\mathcal{C}}^-$, respectively.

The operator D in (2.1) can then be separated in the operators

$$D^{\psi^-} g = \frac{1}{(2\pi)^4} \sum_j \varphi_j \int \sigma(D)(x,\xi)\psi^-(\xi)\widehat{\chi_j g}(\xi) d\xi$$
$$D^{\psi^0} g = \frac{1}{(2\pi)^4} \sum_j \varphi_j \int \sigma(D)(x,\xi)\psi^0(\xi)\widehat{\chi_j g}(\xi) d\xi$$
$$D^{\psi^+} g = \frac{1}{(2\pi)^4} \sum_j \varphi_j \int \sigma(D)(x,\xi)\psi^+(\xi)\widehat{\chi_j g}(\xi) d\xi.$$

We further use the notation g^{ψ^-} to denote the function given by

$$g^{\psi^-} = \Psi^- g$$

with similar meanings for g^{ψ^0} and g^{ψ^+}. Then $D^{\psi^-}g$ can be expressed in terms of g^{ψ^-} by

$$D^{\psi^-}g = \frac{1}{(2\pi)^4}\sum_j \varphi_j \int \sigma(D)(x,\xi)\psi^-(\xi)\widehat{\chi_j g}(\xi)d\xi.$$

We generally drop the cutoffs χ_j, keeping in mind we work locally, writing, for instance

$$D^{\psi^-}g = Dg^{\psi^-}$$

by which we mean the above sum, modulo smooth terms.

We explain here briefly the idea used to solve the boundary equations which arise with the use of microlocalizations. In finding solutions to equations involving pseudodifferential operators we will take into account the behavior of the symbol of the operators in the different regions $\widetilde{\mathcal{C}}^-$, $\mathcal{C}^0$ and $\widetilde{\mathcal{C}}^+$. Thus, to find a u which satisfies

$$Du = f,$$

or

$$\frac{1}{(2\pi)^4}\sum_j \varphi_j \int \sigma(D)(x,\xi)\widehat{\chi_j u}(\xi)d\xi = \sum_j \varphi_j f,$$

we look to solve the equation locally, to find u supported in $\{\chi_j \equiv 1\}$ such that

$$\frac{1}{(2\pi)^4}\int \sigma(D)(x,\xi)\widehat{\chi_j u}(\xi)d\xi = f$$

in a neighborhood of $\varphi_j \equiv 1$. We then look to solve the equation with three components to f:

$$\varphi_j f = (\varphi_j f)^{\psi^-} + (\varphi_j f)^{\psi^0} + (\varphi_j f)^{\psi^+},$$

where $(\varphi_j f)^{\psi^-}$ (resp. $(\varphi_j f)^{\psi^0}$ and $(\varphi_j f)^{\psi^+}$) has a transform supported in $\widetilde{\mathcal{C}}^-$ (resp. $\mathcal{C}^0$ and $\widetilde{\mathcal{C}}^+$). A solution to $Du = f$ can possibly (depending on the operator D) be found by setting $u = u^- + u^- + u^+$ where u^- (resp. u^0 and u^+) solves $Du^- = f^{\psi^-}$ (resp. $Du^0 = f^{\psi^0}$ and $Du^+ = f^{\psi^+}$. The advantage of the three separate equations is that we can consider separately how $\sigma(D)$ behaves on $\widetilde{\mathcal{C}}^-$, $\mathcal{C}^0$ and $\widetilde{\mathcal{C}}^+$. For instance, to find an expression for an approximate solution, u_a^-, which satisfies $Du_a^- = f^{\psi^-} + \Psi^{-1}f$ we look for a pseudodifferential operator, $D^{-1}_{\mathcal{C}^-}$, such that

$$D \circ D^{-1}_{\mathcal{C}^-} = \widetilde{\Psi^-},$$

where $\widetilde{\Psi^-}$ is a 0 order pseudodifferential operator whose symbol is equivalently 1 on the support of ψ^-, modulo Ψ^{-1}. The idea is then that if the symbol of the operator D behaves in such a way on $\mathcal{C}^-$ so as to give rise to a pseudodifferential operator $D^{-1}_{\mathcal{C}^-}$ with the above property, the operator

$D^{-1}_{\mathcal{C}^-}$ is a type of inverse for D. The solution u_a^- can then be expressed as

$$u_a^- = D^{-1}_{\mathcal{C}^-} f^{\psi^-}.$$

Note that modulo smooth terms the support of $\widehat{u_a^-}(\xi)$ is contained in $\widetilde{\mathcal{C}}^-$.

This procedure will be used in Section 4, where there will be a combination of operators to invert (or otherwise eliminate) others acting on a sought after solution.

3. The boundary value problem

We follow [2] in setting up our boundary value problem (which is similar to the setup of the $\bar\partial$-Neumann problem in [2]). We let ρ denote the geodesic distance (with respect to the standard Euclidean metric) to the boundary function for $\Omega \subset \mathbb{C}^2$, a smoothly bounded pseudoconvex domain. We let U be an open neighborhood of $\partial\Omega$ such that

$$\begin{aligned} &\Omega \cap U = \{z \in U | \rho(z) < 0\};\\ &\nabla\rho(z) \neq 0 \qquad \text{for } z \in U. \end{aligned}$$

We define an orthonormal frame of $(1,0)$-forms on a neighborhood U with ω_1, ω_2 where $\omega_2 = \sqrt{2}\partial\rho$, and L_1, L_2 the dual frame. We thus can write

$$\begin{aligned} L_1 &= \frac{1}{2}(X_1 - iX_2) + O(\rho)\\ L_2 &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial\rho} + iT + O(\rho) \end{aligned} \tag{3.1}$$

where $\partial/\partial\rho$ is the vector field dual to $d\rho$, and X_1, X_2, and T are tangential fields.

We denote by R the operator which restricts to the boundary ($\rho = 0$). We then choose coordinates (x_1, x_2, x_3) on $\partial\Omega$ near a point $p \in \partial\Omega$, in terms of which the vector fields $R \circ L_1$ and $T_b := R \circ T$ are given by

$$\begin{aligned} T_b &= \frac{\partial}{\partial x_3}\\ L_{b1} := R \circ L_1 &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) + \sum_{j=1}^{3} \ell_j(x)\frac{\partial}{\partial x_j}, \end{aligned} \tag{3.2}$$

where $\ell_j(x) = O(x - p)$ for $j = 1, 2, 3$.

The pseudodifferential operators used in this article will be implicitly defined with a family of cutoffs χ_j as in (2.1) in the support of which we can find coordinates (x, ρ) as above. The tangent coordinates x, their dual coordinates, ξ, and with these, the regions $\widetilde{\mathcal{C}}^-$, $\mathcal{C}^0$, and $\widetilde{\mathcal{C}}^+$ are defined locally with respect to the these sets of coordinates above.

We also define the scalar function s by

$$\bar\partial\bar\omega_1 = s\bar\omega_1 \wedge \bar\omega_2.$$

Following [2], we use a Green's operator and Poisson operator to reduce the following boundary value problem to the boundary:

$$\Box' u = f, \tag{3.3}$$

where $\Box' = \vartheta\bar{\partial}' + \bar{\partial}\vartheta$, and $\bar{\partial}'$ is the operator on $(0,1)$-forms given by

$$\bar{\partial}'(u_1\bar{\omega}_1 + u_2\bar{\omega}_2) = \bar{\partial}u + \Phi^0(u)\bar{\omega}_1 \wedge \bar{\omega}_2,$$

where Φ^0 is a zero order operator defined later, with the boundary conditions

$$\overline{L}_2 u_1 - s_0 u_1 - \overline{L}_1 u_2 + \Phi_b^0(u) = 0, \tag{3.4}$$

modulo $\Psi^{-1}(\Omega)$ (resp. $\Psi_b^{-1}(\partial\Omega)$) terms.

The operator Φ^0 will be chosen to have a symbol which is independent of ρ and its transform variable η, and so Φ_b^0 is the same operator, simply restricted to boundary forms. We write $\Phi^0(u_1\bar{\omega}_1 + u_2\bar{\omega}_2) = \Phi_1^0(u_1) + \Phi_2^0(u_2)$. The operators Φ_1^0 and Φ_2^0 are given later, according to the symbols, $\phi_1(x,\xi)$ and $\phi_2(x,\xi)$, defined in (4.5) and (4.7), respectively.

Thus the boundary condition, (3.4), becomes

$$\overline{L}_2 u_1 - s_0 u_1 - \overline{L}_1 u_2 + \Phi_1^0(u_1) + \Phi_2^0(u_2) = 0$$

on $\partial\Omega$.

A solution to (3.3) and (3.4) can be written in terms of a Green's and a Poisson's operator. The Green's operator

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

solves

$$\begin{aligned} 2\Box' \circ G &= I \qquad \text{on } \Omega \\ R \circ G &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

modulo smooth terms, where R is the restriction to the boundary operator as above. If $f = f_1\bar{\omega}_1 + f_2\bar{\omega}_2$, we write

$$G(f) = G_1(f)\bar{\omega}_1 + G_2(f)\bar{\omega}_2,$$

where

$$\begin{aligned} G_1(f) &= G_{11}(f_1) + G_{12}(f_2) \\ G_2(f) &= G_{21}(f_1) + G_{22}(f_2). \end{aligned}$$

And P is a Poisson's operator for the boundary value problem

$$\begin{aligned} 2\Box' \circ P = 0 &\qquad \text{on } \Omega \\ R \circ P = I &\qquad \text{on } \partial\Omega, \end{aligned}$$

modulo smooth terms. With the solution u written $u = u_1\bar{\omega}_1 + u_2\bar{\omega}_2$, we write its restriction to $\partial\Omega$ as

$$u_b = u_b^1\bar{\omega}_1 + u_b^2\bar{\omega}_2.$$

A solution to (3.3) under condition (3.4) is then given by

$$u = G(2f) + P(u_b). \tag{3.5}$$

We use the notation here and in what follows that u_b is to be understood as $u|_{\partial\Omega} \times \delta(\rho)$ for the defining function ρ when we write u_b in combination with a pseudodifferential operator on Ω. Otherwise u_b will denote simply the boundary form $u|_{\partial\Omega}$.

The Poisson's operator is also a matrix of operators:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and we further isolate the first and second components of $P(u_b)$ according to

$$P(u_b) = P_1(u_b)\bar{\omega}_1 + P_2(u_b)\bar{\omega}_2,$$

so that

$$\begin{aligned} P_1(u_b) =& P_{11}(u_b^1) + P_{12}(u_b^2) \\ P_2(u_b) =& P_{21}(u_b^1) + P_{22}(u_b^2) \end{aligned}$$

hold.

In what follows, we will need descriptions for operators given by restrictions to the boundary of a normal derivative applied to G_1 and to P_1. These are calculated in [5]. We use the symbol $\Xi(x, \xi)$ given by

$$\begin{aligned} \Xi^2(x,\xi) =& 2\sigma(L_{b1})\sigma(\overline{L}_{b1}) + 2\xi_3^2 \\ =& 2\left|\frac{1}{2}(\xi_1 + i\xi_2) + \sum_j \ell_j\xi_j\right|^2 + 2\xi_3^2, \end{aligned}$$

where the functions $\ell_j(x)$ come from (3.2). We can also use the matrix E to describe the symbol $\Xi(x,\xi)$:

$$E = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} + 2\begin{bmatrix} \mathrm{Re}\,\ell_1 & \mathrm{Re}\,\ell_2 & \mathrm{Re}\,\ell_3 \\ -\mathrm{Im}\,\ell_1 & -\mathrm{Im}\,\ell_2 & -\mathrm{Im}\,\ell_3 \\ 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} \overline{\ell}_1\ell_1 & \overline{\ell}_2\ell_1 & \overline{\ell}_3\ell_1 \\ \overline{\ell}_1\ell_2 & \overline{\ell}_2\ell_2 & \overline{\ell}_3\ell_2 \\ \overline{\ell}_1\ell_3 & \overline{\ell}_2\ell_3 & \overline{\ell}_3\ell_3 \end{bmatrix}.$$

Thus

$$\Xi^2(x,\xi) = \xi^t E \xi$$

holds.

Recall that R denotes the restriction to the boundary. For a normal derivative applied to G we have

Theorem 3.1. *Let $\Theta \in \Psi^{-1}(\Omega)$ be the operator with symbol*

$$\sigma(\Theta) = \frac{i}{\eta - i|\Xi(x,\xi)|}.$$

Then modulo smooth terms

$$R\frac{\partial}{\partial\rho} \circ G(g) = R \circ \Theta(g) + R \circ \Psi^{-2}g. \tag{3.6}$$

The operator Θ is to be understood as a diagonal matrix of operators with symbols given by $\frac{i}{\eta - i|\Xi(x,\xi)|}$.

The operator given by the restriction to the boundary of the inward normal derivative of the solution to a Dirichlet problem, i.e. $R \circ \frac{\partial}{\partial\rho}P(u_b)$, is termed the Dirichlet to Neumann operator (DNO). We denote the DNO as N:

$$N := R \circ \frac{\partial}{\partial\rho}P.$$

We use the following notations. The Kohn Laplacian, $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, in terms of normal and tangential derivatives (locally) can be written as

$$2\Box = -\frac{\partial^2}{\partial\rho^2} + C\frac{\partial}{\partial\rho} + \Box_\rho,$$

where

$$C = \sqrt{2}s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(see [2]). The values of function $s(x,\rho)$ restricted to the boundary will play a role in the DNO, and we denote this by $s_0(x) := s(x,0)$.

$\Box_\rho$ is a second order tangential operator. We restrict to the boundary the first order tangential derivatives contained in $\Box_\rho$ and denote the resulting boundary operator using the coefficient $\alpha_0^j(x)$, the subscript 0 being used to remind us of the restriction to the boundary; $\alpha_0^j(x)$ is a matrix

of functions (coefficients of the first order tangential derivatives) such that

$$R\left(\sigma_1(\Box_\rho)\right) = \sum_j \alpha_0^j(x)\xi_j.$$

We collect the second order terms in $\sigma(\Box_\rho)$ which are $O(\rho)$, with the coefficients $\tau_0^{jk}(x)$:

$$R \circ \frac{\partial}{\partial\rho}\sigma_2(\Box_\rho) = -\sum_{j,k=1}^{3} \tau_0^{jk}(x)\xi_j\xi_k.$$

Then from [5], we have the description of the highest order terms for the DNO:

Theorem 3.2. *Let N denote the Dirichlet to Neumann operator for a Dirichlet problem with the operator $2\Box'$. Then the first two highest order terms in the symbol expansion for N are given as follows:*

$$\begin{aligned}
\sigma_1(N)(x,\xi) =&|\Xi(x,\xi)|,\\
\sigma_0(N)(x,\xi) =&\frac{i}{8}\frac{\sum_{j=1}^{3}\left(\xi^t(\partial_{x_j}E)\xi\right)\left(e_j^t E\xi + \xi^t E e_j\right)}{|\Xi(x,\xi)|^3}\\
&+\frac{\sqrt{2}}{2}s_0(x) + \frac{1}{2}\frac{\sum_{j=1}^{3}\alpha_0^j(x)\xi_j}{|\Xi(x,\xi)|} + \frac{1}{4}\frac{\tau_0^{jk}(x)\xi_j\xi_k}{\Xi^2(x,\xi)}\\
&+\frac{\sqrt{2}}{2}\begin{bmatrix}\phi_1(x,\xi) & \phi_2(x,\xi)\\ 0 & 0\end{bmatrix} - \frac{1}{|\Xi(x,\xi)|}\begin{bmatrix}\xi_3\phi_1(x,\xi) & \xi_3\phi_2(x,\xi)\\ \sigma(L_1)\phi_1(x,\xi) & \sigma(L_1)\phi_2(x,\xi)\end{bmatrix}.
\end{aligned}$$

We write

$$R \circ \frac{\partial}{\partial\rho}P_1(u_b) = N_1 u_b^1 + N_2 u_b^2$$

so that N_1 is actually the $(1,1)$ entry of the DNO matrix operator and N_2 the $(1,2)$ entry. Thus,

$$\begin{aligned}
\sigma_1(N_1) =&\sigma_1(N_1^0) = |\Xi(x,\xi)|\\
\sigma_0(N_1) =&\sigma_0(N_1^0) + \phi_1(x,\xi)\left(\frac{\sqrt{2}}{2} - \frac{\xi_3}{|\Xi(x,\xi)|}\right),
\end{aligned}$$

where N_1^0 is the $(1,1)$ entry of the DNO matrix in the case $\phi_1 = \phi_2 = 0$, and corresponds to the DNO operator in [2]. Similarly,

$$\begin{aligned}
\sigma_1(N_2) =&0\\
\sigma_0(N_2) =&\frac{1}{2}\frac{\sum_{j=1}^{3}\alpha_{0,12}^j(x)\xi_j}{|\Xi(x,\xi)|} + \phi_2(x,\xi)\left(\frac{\sqrt{2}}{2} - \frac{\xi_3}{|\Xi(x,\xi)|}\right),
\end{aligned} \tag{3.7}$$

where $\alpha_{0,lm}^j(x)$ is given by the $(1,2)$ entry of the matrix of symbols $\alpha_0^j(x)$ (defined as in Theorem 3.2).

We can write condition (3.4) locally as

$$\begin{aligned}0 =& R\circ\left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial\rho}-iT_b\right)u^1 - s_0u_b^1-\overline{L}_{b1}u_b^2+\Phi_b^0u_b\\ =& R\circ\left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial\rho}-iT_b\right)\left(G_1(2f)+P_1(u_b)\right)-s_0u_b^1-\overline{L}_{b1}u_b^2+\Phi_b^0u_b\\ =& \frac{1}{\sqrt{2}}R\circ\Theta(2f_1)+R\circ\Psi^{-2}f+\left(\frac{1}{\sqrt{2}}N_1-iT_b\right)u_b^1-s_0u_b^1-\overline{L}_{b1}u_b^2+\frac{1}{\sqrt{2}}N_2u_b^2+\Phi_b^0u_b,\end{aligned}$$

modulo smooth terms, using Theorems 3.1 and 3.2 in the last line, see also [5]. We rewrite this as

$$\left(\frac{1}{\sqrt{2}}N_1-iT_b\right)u_b^1-s_0u_b^1-\overline{L}_{b1}u_b^2+\frac{1}{\sqrt{2}}N_2u_b^2+\Phi_b^0u_b=-\frac{2}{\sqrt{2}}R\circ\Theta f_1+R\circ\Psi^{-2}f,$$

modulo smooth terms.

4. Boundary solution

Our approximate solution u, will be determined via (3.5) by its boundary values. The idea is to choose u_b^1 and u_b^2 which will satisfy the boundary condition above:

$$\left(\frac{1}{\sqrt{2}}N_1-iT_b\right)u_b^1-s_0u_b^1-\overline{L}_{b1}u_b^2+\frac{1}{\sqrt{2}}N_2u_b^2+\Phi_b^0u_b=-\frac{2}{\sqrt{2}}R\circ\Theta f_1+R\circ\Psi^{-2}f, \tag{4.1}$$

modulo (sufficiently smooth) error terms.

We stress here the local nature of Equation 4.1. The boundary relation is a sum of local operators, restricted to the boundary, and so should be read as

$$\begin{aligned}\sum_j\left[\varphi_j\left(\frac{1}{\sqrt{2}}N_1-iT_b\right)(\chi_ju_b^1)-s_0(\chi_ju_b^1)-\varphi_j\overline{L}_{b1}(\chi_ju_b^2)+\varphi_j\frac{1}{\sqrt{2}}N_2(\chi_ju_b^2)+\varphi_j\Phi_b^0(\chi_ju_b)\right]\\ =-\frac{2}{\sqrt{2}}\sum_j R\circ\varphi_j\Theta(\chi_jf_1),\end{aligned}$$

where$\chi_j\equiv 1$ is a covering of Ω (hence $\chi_j\big|_{\partial\Omega}\equiv 1$ forms a covering of $\partial\Omega$), and φ_j is a partition of unity subordinate to this covering. The relation is modulo $R\circ\Psi^{-2}f$, $\Psi_b^{-2}u_b$, and smoothing terms, where, with slight abuse of notation, φ_j and χ_j are as above, but restricted to the boundary. The symbols are then defined locally as in Section 3.

We momentarily assume u_b is supported in supp χ_j and seek a solution to the above equation for a fixed j. We further decompose u_b according to

$$\begin{aligned}u_b^1&=u_b^{1,-}+u_b^{1,0}+u_b^{1,+}\\ u_b^2&=u_b^{2,-}+u_b^{2,0}+u_b^{2,+},\end{aligned}$$

where $u_b^{j,-}$ (resp. $u_b^{j,0}$ and $u_b^{j,+}$) is such that $\widehat{u_b^{j,0}}(\xi)$ is supported in $\widetilde{\mathcal{C}}^-$ (resp. $\mathcal{C}^0$ and $\widetilde{\mathcal{C}}^+$).

We recall from Section 2 the use of superscripts ψ^- to denote operation with the operator Ψ^-. This use of symbol as superscript will have similar meanings with other symbols, for instance, ψ^0 and ψ^+. Whereas, in combination with operators, the superscripts ψ^-, ψ^0, and ψ^+ are to be understood to denote an operator whose symbol is cutoff with the respective ψ function. For example,

$$\sigma\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)^{\psi^-} = \psi^-(\xi)\sigma\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)(x,\xi)$$

and so forth.

We thus look to solve three separate equations stemming from (4.1), namely

$$\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)u_b^{1,-} - s_0 u_b^{1,-} - \overline{L}_{b1}u_b^{2,-} + \frac{1}{\sqrt{2}}N_2 u_b^{2,-} + \Phi_b^0 u_b^- = -\frac{2}{\sqrt{2}}\left(R\circ\Theta f_1\right)^{\psi^-},$$

modulo error terms, with similar equations for $u_b^{j,0}$ and $u_b^{j,+}$, for $j=1,2$.

We expand the highest order term of the operator $\frac{1}{\sqrt{2}}N_1 - iT_b$ in the region $\widetilde{\mathcal{C}}^-$:

$$\begin{aligned}\sigma_1\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right) =& \frac{1}{\sqrt{2}}|\Xi(x,\xi)| + \xi_3\\ =& \sqrt{\xi_3^2 + \left|\frac{1}{2}(\xi_1 + i\xi_2) + \sum_j \ell_j\xi_j\right|^2} + \xi_3\\ =& |\xi_3|\sqrt{1+\kappa} + \xi_3,\end{aligned}$$

where

$$\begin{aligned}\kappa =& \frac{\left|\frac{1}{2}(\xi_1 + i\xi_2) + \sum_j \ell_j\xi_j\right|^2}{\xi_3^2}\\ =& \frac{\sigma(\overline{L}_{b1})\sigma(L_{b1})}{\xi_3^2}.\end{aligned}$$

We impose the condition that $u_b^{1,-}$ is of the form $u_b^{1,-} = g^{\psi^-}$ for some function g (see (4.8) below). Since $\psi^-(\xi)$ has the property supp $\psi^-\big|_{|\xi|=1}$ is a compact subset of $\mathcal{C}^-\cap\{|\xi|=1\}$, we can find a $\widetilde{\psi^-}(\xi)\in C^\infty(\widetilde{\mathcal{C}}^-)$ with the property $\widetilde{\psi^-}(\xi)\big|_{|\xi|=1}$ has compact support in $\mathcal{C}^-\cap\{|\xi|=1\}$, $\widetilde{\psi^-}(\xi) = \widetilde{\psi^-}(\xi/|\xi|)$ for $|\xi|\geq 1$, $\widetilde{\psi^-}\big|_{\{|\xi|\leq 1\}}$ has support which is relatively compact in the interior of the region $\widetilde{\mathcal{C}}^-$, and $\widetilde{\psi^-}\equiv 1$ on supp ψ^-.

Then we can expand the above symbol for $\frac{1}{\sqrt{2}}N_1 - iT_b$ in terms of κ in a small enough conic neighborhood, U, of $(0, \text{supp}\,\widetilde{\psi^-})$. In the conic neighborhood U, $\kappa < c$ for some $c < 1$ and we can write

$$\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)u_b^{1,-} = \left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)^{\widetilde{\psi^-}} u_b^{1,-},$$

modulo smoothing terms, with

$$\sigma_1\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)^{\widetilde{\psi^-}} \begin{aligned}[t] &=\widetilde{\psi^-}(\xi)\left(|\xi_3|\sqrt{1+\kappa}+\xi_3\right)\\ &=\widetilde{\psi^-}(\xi)\left(|\xi_3|\left(1+\frac{1}{2}\kappa-\frac{1}{8}\kappa^2+\cdots\right)+\xi_3\right)\\ &=\widetilde{\psi^-}(\xi)|\xi_3|\left(\frac{1}{2}\kappa-\frac{1}{8}\kappa^2+\cdots\right)\\ &=\widetilde{\psi^-}(\xi)\sigma(\overline{L}_{b1})\frac{\sigma(L_{b1})}{|\xi_3|}\left(\frac{1}{2}-\frac{1}{8}\kappa+\cdots\right).\end{aligned}$$

Since in the neighborhood U the infinite sum in parentheses converges uniformly, and as $(\widetilde{\psi^-})\kappa \in \mathcal{S}^0(\partial\Omega)$, we see that by differentiating the power series the symbol given by

$$\sigma(B_0) = \widetilde{\psi^-}(\xi)\frac{\sigma(L_{b1})}{|\xi_3|}\left(\frac{1}{2}-\frac{1}{8}\kappa+\cdots\right) \tag{4.2}$$

defines an operator $B_0 \in \Psi^0(\partial\Omega)$.

We note that

$$\sigma\left(\overline{L}_{b1}\right)\sigma(B_0) = \sigma\left(\overline{L}_{b1}\circ B_0\right) + c_0(x,\xi)$$

for some $c_0(x,\xi) \in \mathcal{S}^0$, of the form $c_0'(x,\xi)\widetilde{\psi^-}(\xi)$. And so we write

$$\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)u_b^{1,-} = \overline{L}_{b1}\circ B_0(u_b^{1,-}) + \frac{1}{\sqrt{2}}A_0(u_b^{1,-}) + C_0(u_b^{1,-}),$$

where $C_0 = Op(c_0)$, and where $A_0 = Op(\sigma_0(N_1))$ is the 0 order operator with symbol given by the 0th order term in the symbol expansion of N_1.

We next separate out the terms involving the symbols $\phi_1(x,\xi)$ from $\sigma(A_0)$, and write

$$\frac{1}{\sqrt{2}}\sigma(A_0) = \frac{1}{\sqrt{2}}n_{11}^0(x,\xi) + \frac{\phi_1(x,\xi)}{2}\left(1-\sqrt{2}\frac{\xi_3}{|\Xi(x,\xi)|}\right), \tag{4.3}$$

where $n_{11}^0(x,\xi)$ is the 0th order symbol in the $(1,1)$ entry of $\sigma_0(N)$, modulo ϕ terms in Theorem 3.2, i.e., the $(1,1)$ entry of $\sigma_0(N^0)$, where N^0 is the DNO calculated according to $\Phi \equiv 0$ as in [2].

From the left hand side of (4.1) we first look at

$$\begin{aligned}\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)u_b^{1,-} - s_0u_b^{1,-} &- \overline{L}_{b1}u_b^{2,-} + \frac{1}{\sqrt{2}}N_2u_b^{2,-} + \Phi_b^0u_b^-\\ &= \overline{L}_{b1}\circ B_0(u_b^{1,-}) + \left(\frac{1}{\sqrt{2}}A_0 + C_0 - s_0 + \Phi_{1b}^0\right)u_b^{1,-}\\ &\quad - \overline{L}_{b1}(u_b^{2,-}) + \frac{1}{\sqrt{2}}N_2u_b^{2,-} + \Phi_{2b}^0u_b^{2,-}.\end{aligned} \tag{4.4}$$

From (4.3) we have

$$\sigma\left(\frac{1}{\sqrt{2}}A_0+C_0-s_0+\Phi^0_{1b}\right)=\frac{1}{\sqrt{2}}n^0_{11}(x,\xi)+c_0(x,\xi)-s_0(x)+\frac{\phi_1(x,\xi)}{2}\left(3-\sqrt{2}\frac{\xi_3}{|\Xi(x,\xi)|}\right).$$

In $\widetilde{\mathcal{C}}^-$ the factor $3-\sqrt{2}\frac{\xi_3}{|\Xi(x,\xi)|}$ satisfies

$$3-\sqrt{2}\frac{\xi_3}{|\Xi(x,\xi)|}\geq 4.$$

We thus take a cutoff function (symbol of class order 0) φ such that $\varphi(\xi)\in C^\infty(\widetilde{\mathcal{C}}^-)$ with similar properties in relation to $\widetilde{\psi^-}$, which $\widetilde{\psi^-}$ had in relation to ψ^-: $\varphi(\xi)\in C^\infty(\widetilde{\mathcal{C}}^-)$ with the property $\varphi(\xi)\big|_{|\xi|=1}$ has compact support in $\mathcal{C}^-\cap\{|\xi|=1\}$, $\varphi(\xi)=\varphi(\xi/|\xi|)$ for $|\xi|\geq 1$, $\varphi\big|_{\{|\xi|\leq 1\}}$ has support which is relatively compact in the interior of the region $\widetilde{\mathcal{C}}^-$, and $\varphi\equiv 1$ on supp $\widetilde{\psi^-}$. Then we set

$$\phi_1(x,\xi):=2\varphi(\xi)\frac{1+s_0(x)-\frac{1}{\sqrt{2}}n^0_{11}(x,\xi)-c_0(x,\xi)}{3-\sqrt{2}\frac{\xi_3}{|\Xi(x,\xi)|}}. \tag{4.5}$$

It is easy to check that ϕ_1 defines a symbol in class $\mathcal{S}^0(\partial\Omega)$. With this choice of ϕ_1 (4.4) leads us to consider

$$\overline{L}_{b1}\circ B_0(u^{1,-}_b)+u^{1,-}_b-\overline{L}_{b1}(u^{2,-}_b)+\frac{1}{\sqrt{2}}N_2u^{2,-}_b+\Phi^0_{2b}u^{2,-}_b=-\frac{2}{\sqrt{2}}\left(R\circ\Theta f_1\right)^{\psi^-}. \tag{4.6}$$

From (3.7) we have

$$\begin{aligned}\sigma\left(\frac{1}{\sqrt{2}}N_2+\Phi^0_{2b}\right)&=\frac{1}{2\sqrt{2}}\frac{\sum_{j=1}^3\alpha^j_{0,12}(x)\xi_j}{|\Xi(x,\xi)|}+\frac{\phi_2(x,\xi)}{2}\left(1-\sqrt{2}\frac{\xi_3}{|\Xi(x,\xi)|}\right)+\phi_2(x,\xi)\\&=\frac{1}{2\sqrt{2}}\frac{\sum_{j=1}^3\alpha^j_{0,12}(x)\xi_j}{|\Xi(x,\xi)|}+\frac{\phi_2(x,\xi)}{2}\left(3-\sqrt{2}\frac{\xi_3}{|\Xi(x,\xi)|}\right)\end{aligned}$$

modulo terms in $\mathcal{S}^{-1}(\partial\Omega)$. We thus choose $\phi_2(x,\xi)$ according to

$$\phi_2(x,\xi):=-\frac{\varphi(\xi)}{\sqrt{2}}\frac{\sum_{j=1}^3\alpha^j_{0,12}(x)\xi_j}{3|\Xi(x,\xi)|-\sqrt{2}\xi_3}, \tag{4.7}$$

where φ is as above.

With the choice of $u^{1,-}_b$ according to

$$u^{1,-}_b=-\frac{2}{\sqrt{2}}\left(R\circ\Theta f_1\right)^{\psi^-}, \tag{4.8}$$

we need to choose $u^{2,-}_b$ so that (modulo error terms)

$$\overline{L}_{b1}\circ B_0(u^{1,-}_b)-\overline{L}_{b1}(u^{2,-}_b)=0, \tag{4.9}$$

taking into account the definitions of ϕ_1 and ϕ_2 above.

Since $\widetilde{\psi^-} \equiv 1$ on supp ψ^-, if $\widetilde{\Psi^-}$ is the operator corresponding to the symbol $\widetilde{\psi^-}$, we have

$$B_0 \left(R \circ \Theta f_1\right)^{\psi^-} = \widetilde{\Psi^-} \circ \left(B_0 \circ R \circ \Theta f_1\right)^{\psi^-}$$

modulo $\Psi_b^{-\infty} \circ R \circ \Psi^{-1} f$.

We thus set

$$\begin{aligned} u_b^{2,-} =& \widetilde{\Psi^-} \circ B_0 u_b^{1,-} \\ =& -\frac{2}{\sqrt{2}} B_0 \left(R \circ \Theta f_1\right)^{\psi^-} + \Psi_b^{-\infty} \circ R \circ \Psi^{-1} f \\ =& B_0(u_b^{1,-}) + \Psi_b^{-\infty} \circ R \circ \Psi^{-1} f, \end{aligned} \tag{4.10}$$

so that (4.9) is satisfied, modulo $\Psi_b^{-\infty} \circ R \circ \Psi^{-1} f$.

It follows that $u_b^{1,-}$ and $u_b^{2,-}$ can be written as

$$\begin{aligned} u_b^{1,-} =& \Psi_b^0 \circ R \circ \Psi^{-1} f \\ u_b^{2,-} =& \Psi_b^0 \circ R \circ \Psi^{-1} f \end{aligned} \tag{4.11}$$

as $\psi^-(\xi)$ is a (boundary) symbol of order 0, B_0 a (boundary) operator of order 0, and Θ an operator of order -1.

We next seek a combination of the terms which solves

$$\left(\frac{1}{\sqrt{2}} N_1 - i T_b\right) u_b^{1,+} + \Psi^0 u_b^+ - \overline{L}_{b1} u_b^{2,+} = -\frac{2}{\sqrt{2}} \left(R \circ \Theta f_1\right)^{\psi^+}, \tag{4.12}$$

where Ψ^0 stands for the operators of order 0 on the left hand side of (4.1).

In $\widetilde{\mathcal{C}}^+$ we have

$$\begin{aligned} \sigma_1\left(\frac{1}{\sqrt{2}} N_1 - i T_b\right) =& \frac{1}{\sqrt{2}} |\Xi(x,\xi)| + \xi_3 \\ \gtrsim& |\xi|, \end{aligned}$$

and since there exists a $c > 0$ such that $\xi_3 > c$ in supp ψ^+, we can invert the operator $\frac{1}{\sqrt{2}} N - i T_b$. With this in mind we define the symbol

$$\begin{aligned} \alpha^{\widetilde{\psi^+}}(x,\xi) =& \frac{\widetilde{\psi^+}(\xi)}{\sigma_1\left(\frac{1}{\sqrt{2}} N_1 - i T_b\right)} \\ =& \frac{\widetilde{\psi^+}(\xi)}{\frac{1}{\sqrt{2}} |\Xi(x,\xi)| + \xi_3}, \end{aligned}$$

where $\widetilde{\psi^+}$ is defined in analogy to $\widetilde{\psi^-}$ above. Namely, $\widetilde{\psi^+}$ has the properties $\widetilde{\psi^+}(\xi) \in C^\infty(\widetilde{\mathcal{C}}^+)$, $\widetilde{\psi^+}(\xi) = \widetilde{\psi^+}(\xi/|\xi|)$ for $|\xi| \geq 1$, and such that $\widetilde{\psi^+} \equiv 1$ on supp ψ^+. Also, the restriction, $\psi_D^+\big|_{\{|\xi| \leq 1\}}$, has relatively compact support in in the interior of $\widetilde{\mathcal{C}}^+$.

The operator, $Op(\alpha^{\widetilde{\psi^+}})$ then behaves as a type of inverse to the operator $\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)$ (in the region $\widetilde{\mathcal{C}}^+$):

$$\begin{aligned}\sigma\left[\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)\circ Op(\alpha^{\widetilde{\psi^+}})\right] =&\sigma\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)\sigma(\alpha^{\widetilde{\psi^+}})\\ =&\left(\frac{1}{\sqrt{2}}|\Xi(x,\xi)|+\xi_3\right)\frac{\widetilde{\psi^+}(\xi)}{\frac{1}{\sqrt{2}}|\Xi(x,\xi)|+\xi_3}\\ =&\widetilde{\psi^+}(\xi)\end{aligned}$$

modulo $\mathcal{S}^{-1}(\partial\Omega)$. Furthermore, the same calculations give

$$\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)^{\psi^+}\circ Op(\alpha^{\widetilde{\psi^+}}) = \Psi^+$$

modulo $\Psi^{-1}(\partial\Omega)$.

We thus choose $u_b^{1,+}$ according to

$$u_b^{1,+} = \left[Op(\alpha^{\widetilde{\psi^+}})\left(-\frac{2}{\sqrt{2}}R\circ\Theta f_1\right)\right]^{\psi^+}. \tag{4.13}$$

Then, from above,

$$\begin{aligned}\left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)u_b^{1,+} =& \left(\frac{1}{\sqrt{2}}N_1 - iT_b\right)^{\psi^+}\circ Op(\alpha^{\widetilde{\psi^+}})\left(-\frac{2}{\sqrt{2}}R\circ\Theta f_1\right)\\ =&\left(-\frac{2}{\sqrt{2}}R\circ\Theta f_1\right)^{\psi^+} + \Psi_b^{-1}\circ R\circ\Psi^{-1}f.\end{aligned}$$

Then with $u_b^{1,+}$ according to (4.13) and with $u_b^{2,+}=0$, (4.12) is satisfied, modulo error terms of the form $\Psi_b^{-1}\circ R\circ\Psi^{-1}f$.

Furthermore, we have

$$\begin{aligned}u_b^{1,+} =&\Psi_b^{-1}\circ R\circ\Psi^{-1}f\\ u_b^{2,+} =&R\circ\Psi^{-\infty}f.\end{aligned} \tag{4.14}$$

In the region $\mathcal{C}^0$ we can find an operator which acts as an inverse to $\overline{L}_{b1}$ since

$$\sigma(\overline{L}_{b1}) \gtrsim |\xi_1 + i\xi_2| \gtrsim |\xi|.$$

Hence, the choice for $u_b^{1,0}$ and $u_b^{2,0}$ is analogous (but reversed) to the case of $u_b^{1,+}$ and $u_b^{2,+}$ above. Namely, we take $u_b^{1,0}=0$ and $u_b^{2,0}$ to be given by

$$u_b^{2,0} := \left[Op(\beta^{\widetilde{\psi^0}})\left(\frac{2}{\sqrt{2}}R\circ\Theta f_1\right)\right]^{\psi^0}, \tag{4.15}$$

where

$$\beta^{\widetilde{\psi^0}}(x,\xi)=\frac{\widetilde{\psi^0}(\xi)}{\sigma(\overline{L}_{b1})},$$

and $\widetilde{\psi^0}(\xi)$ is defined analogously to $\widetilde{\psi^-}(\xi)$ and $\widetilde{\psi^+}(\xi)$ above.

With $u_b^{1,0}$ and $u_b^{2,0}$ so chosen, we have

$$\left(\frac{1}{\sqrt{2}}N_1-iT_b\right)u_b^{1,0}+\Psi^0u_b^0-\overline{L}_{b1}u_b^{2,0}=-\frac{2}{\sqrt{2}}\left(R\circ\Theta f_1\right)^{\psi^0},$$

and

$$\begin{aligned} u_b^{1,0} &= R\circ\Psi^{-\infty}f \\ u_b^{2,0} &= \Psi_b^{-1}\circ R\circ\Psi^{-1}f. \end{aligned} \tag{4.16}$$

We collect here our solutions to the boundary problems, u_b^1 and u_b^2. From (4.8), (4.10), (4.13), and (4.15), we have

$$\begin{aligned} u_b^1 &= -\frac{2}{\sqrt{2}}\left(R\circ\Theta f_1\right)^{\psi^-}+\left[Op(\alpha^{\widetilde{\psi^+}})\left(-\frac{2}{\sqrt{2}}R\circ\Theta f_1\right)\right]^{\psi^+} \\ u_b^2 &= -\frac{2}{\sqrt{2}}\widetilde{\Psi^-}\circ B_0\left(R\circ\Theta f_1\right)^{\psi^-}+\left[Op(\beta^{\widetilde{\psi^0}})\left(\frac{2}{\sqrt{2}}R\circ\Theta f_1\right)\right]^{\psi^0}. \end{aligned} \tag{4.17}$$

The expressions in (4.17) were obtained with the assumption u_b is supported in a neighborhood of χ_j, but as written in terms of pseudodifferential operators they automatically translate to global expressions: with φ_j a partition of unity, $u_b=\sum_j\varphi_ju_b$, and

$$\begin{aligned} \varphi_ju_b^1 &= -\frac{2}{\sqrt{2}}\varphi_j\left(R\circ\Psi^-\circ\Theta\right)(\chi_jf_1)-\frac{2}{\sqrt{2}}\varphi_j\left(\Psi^+\circ Op(\alpha^{\widetilde{\psi^+}})\circ R\circ\Theta\right)(\chi_jf_1) \\ \varphi_ju_b^2 &= -\frac{2}{\sqrt{2}}\varphi_j\left(\Psi^-\circ B_0\circ R\circ\Theta\right)(\chi_jf_1)-\frac{2}{\sqrt{2}}\varphi_j\left(\Psi^0\circ Op(\beta^{\widetilde{\psi^0}})\circ R\circ\Theta\right)(\chi_jf_1), \end{aligned}$$

modulo $\Psi_b^0\circ R\circ\Psi^{-\infty}f$, where each operator is supposed to have a symbol as described in earlier sections valid in a neighborhood of supp χ_j.

Then from (4.11), (4.14), and (4.16), we have

$$u_b^1=\Psi_b^0\circ R\circ\Psi^{-1}f \tag{4.18}$$

and as well,

$$u_b^2=\Psi_b^0\circ R\circ\Psi^{-1}f. \tag{4.19}$$

5. Solution operator to $\bar\partial$ with estimates

We now obtain estimates on our solution, with the goal of proving continuity for the solution operator between Sobolev s spaces for all $s \geq 0$. We start with estimates for the solution to the boundary problem in the previous section:

Theorem 5.1. *Let u be defined by* (3.5) *and* (4.17). *Then u satisfies*

$$\Box' u = f \qquad \text{on } \Omega,$$

modulo smooth terms, with the boundary relation

$$\overline{L}_2 u_1 - s_0 u_1 - \overline{L}_1 u_2 + \Phi_b^0 u_b = R \circ \Psi^{-2} f + \Psi_b^{-1} \circ R \circ \Psi^{-1} f. \tag{5.1}$$

Furthermore, we have the estimates

$$\|u_b\|_{W^{s+1/2}(\partial\Omega)} \lesssim \|f\|_{W^s(\Omega)} \tag{5.2}$$

$$\|u\|_{W^{s+1}(\Omega)} \lesssim \|f\|_{W^s(\Omega)} \tag{5.3}$$

for $s \geq 0$.

Proof. For u_b^1 defined as in (4.17) we have estimates from (4.18)

$$\begin{aligned}\|u_b^1\|_{W^{s+1/2}(\partial\Omega)} &\lesssim \|R \circ \Psi^{-1} f\|_{W^{s+1/2}(\partial\Omega)} \\ &\lesssim \|\Psi^{-1} f\|_{W^{s+1}(\Omega)} \\ &\lesssim \|f\|_{W^s(\Omega)}.\end{aligned}$$

Similarly, for u_b^2 defined as in (4.17) we have estimates from (4.19)

$$\|u_b^2\|_{W^{s+1/2}(\partial\Omega)} \lesssim \|f\|_{W^s(\Omega)},$$

and hence (5.2).

Lastly, we recall u as defined on Ω by (3.5):

$$u = G(2f) + P(u_b).$$

We can then use the estimate (5.2) in combination with the regularity properties of the Green's operator and Poisson operator:

$$G : W^s_{(0,1)}(\Omega) \to W^{s+2}_{(0,1)}(\Omega)$$

and

$$P : W^s_{(0,1)}(\partial\Omega) \to W^{s+1/2}_{(0,1)}(\Omega)$$

(these are well-known regularity properties of the Green's and Poission operators, see for instance [5] for proofs) to estimate the terms $G(2f) + P(u_b)$, leading to

$$\begin{aligned}\|u\|_{W^s(\Omega)} \lesssim &\|G(2f) + P(u_b)\|_{W^s(\Omega)}\\ \lesssim &\|f\|_{W^{s-2}(\Omega)} + \|u_b\|_{W^{s-1/2}(\partial\Omega)}\\ \lesssim &\|f\|_{W^{s-2}(\Omega)} + \|f\|_{W^{s-1}(\Omega)}.\end{aligned}$$

□

With the estimates in Theorem 5.1 for the approximate solution to $\Box'$ with the boundary condition $\bar{\partial}'u = 0$, the solution operator to $\bar{\partial}$ in our Main Theorem is constructed almost verbatim as in [5]. We prove the

Theorem 5.2. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Let $f \in W^s_{(0,1)}(\Omega)$ such that $\bar{\partial} f = 0$. Then there exists a solution operator, K, such that*

$$\bar{\partial} K f = f$$

with the property $K : W^s_{(0,1)}(\Omega) \cap ker\, \bar{\partial} \to W^s(\Omega)$ continuously for all $s \geq 0$.

We base our construction of the solution operator on our solution to the boundary value problem

$$\Box' u = f \qquad \text{on } \Omega, \tag{5.4}$$

with the boundary relation

$$\overline{L}_2 u_1 - s_0 u_1 - \overline{L}_1 u_2 + \Phi^0_b u_b = R \circ \Psi^{-2} f + \Psi_b^{-1} \circ R \circ \Psi^{-1} f. \tag{5.5}$$

Theorem 5.1 gave estimates of our chosen solution. In addition we have estimates for $\bar{\partial}'u$:

Lemma 5.3.

$$\|\bar{\partial}' u\|_{W^{s+2}(\Omega)} \lesssim \|f\|_{W^s(\Omega)}.$$

The proof of Lemma 5.3 is contained in [5] and is based on the realization of $\bar{\partial}'u$ as the solution to a Dirichlet problem, of the form $\bar{\partial}\vartheta(\bar{\partial}'u) = 0$, with boundary data satisfying

$$\bar{\partial}'u\big|_{\partial\Omega} = R \circ \Psi^{-2} f + \Psi_b^{-1} \circ R \circ \Psi^{-1} f.$$

Proof of Theorem 5.2. From the definition $\Box' u = \bar{\partial}\vartheta u + \vartheta\bar{\partial}' u$ we have

$$\begin{aligned}\bar{\partial}\left(\vartheta u\right) = &\Box' u - \vartheta\bar{\partial}' u\\ = &f - \vartheta\bar{\partial}' u,\end{aligned} \tag{5.6}$$

modulo smooth terms, for the solution, u, given as in Theorem 5.1. The term $\vartheta\bar{\partial}'u$ can be estimated by Lemma 5.3.

For $\delta > 0$, we let the operator $S_{-\delta} : W^k(\Omega) \to W^{k-\delta}(\Omega)$ be the linear solution operator to

$$\bar\partial v = \vartheta\bar\partial' u \tag{5.7}$$

(with the operators coming from (5.6)) given by Straube in [13] (see Theorem 5.3) (note that from (5.6) it follows that $\vartheta\bar\partial' u$ is $\bar\partial$-closed), i.e., with v defined by

$$v = S_{-\delta}\left(\vartheta\bar\partial' u\right) \tag{5.8}$$

we have (5.7), and

$$\begin{aligned}\|v\|_{W^{s+1-\delta}(\Omega)} &= \left\|S_{-\delta}\left(\vartheta\bar\partial' u\right)\right\|_{W^{s+1-\delta}(\Omega)}\\ &\lesssim \|\vartheta\bar\partial' u\|_{W^{s+1}(\Omega)}\\ &\lesssim \|\bar\partial' u\|_{W^{s+2}(\Omega)}\\ &\lesssim \|f\|_{W^s(\Omega)},\end{aligned}$$

where we use Lemma 5.3 in the last step.

Then, from (5.6), we have the solution $\vartheta u + v$:

$$\bar\partial\left(\vartheta u + v\right) = f \tag{5.9}$$

with estimates

$$\|\vartheta u + v\|_{W^s(\Omega)} \lesssim \|f\|_{W^s(\Omega)}.$$

To write our solution operator, we recall the operators which went into the construction of our solution u. The solution u was written

$$u = P(u_b) + G(2f)$$

where u_b was chosen via (4.17).

We let N' denote the solution operator to the boundary value problem (5.4) and (5.5) given by $N'f = u$, where u and f are as above. Note that N' is a linear operator by construction, and from the estimates from Theorem 5.1, we have in particular

$$N' : W^s_{(0,1)}(\Omega) \to W^{s+1}_{(0,1)}(\Omega)$$

continuously.

Then the solution operator K can be written according to (5.9) as

$$K(f) = \vartheta N'f + S_{-\delta}\left(f - \bar\partial\vartheta N'f\right)$$

As K consists of compositions of linear operators, so is K itself. □

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